A short note on the Vasicek Model

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Abstract

Derivation of the classical Vasicek model from first-principles.

The Vasicek Model.

Vasicek assumed that the instantaneous short rate under the real-world measure evolves as an Ornstein-Uhlenbeck process with constant coefficients. That is:

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t)$$

where k, θ, σ are positive constants.

We exploit the fact, that linear SDEs can be solved using ODE cookbook methods. Consider the ODE:

$$\frac{dr_t}{dt} + kr_t = k\theta$$

This is a linear ODE. The integrating factor is:

$$\begin{aligned} h(t) &= e^{\int k dt} \\ &= e^{kt} \end{aligned}$$

Multiplying the SDE throughout by e^{kt}, we find that:

$$
\begin{aligned}
e^{kt}dr_t + kr_t e^{kt}dt &= k\theta e^{kt}dt + \sigma e^{kt}dW(t) \\
d(e^{kt}r_t) &= k\theta e^{kt}dt + \sigma e^{kt}dW(t) \\
\int_s^t d(e^{kt}r_t) &= k\theta \int_s^t e^{kt}dt + \sigma \int_s^t e^{ku}dW(u) \\
e^{kt}r_t - e^{ks}r_s &= \theta(e^{kt} - e^{ks}) + \sigma \int_s^t e^{ku}dW(u) \\
r_t - e^{-k(t-s)}r_s &= \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)}dW(u) \\
r_t &= e^{-k(t-s)}r_s + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)}dW(u)
\end{aligned}
$$

The Wiener integral $\int_s^t e^{-k(t-u)}dW(u)$ is Gaussian with mean 0 and variance :

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_s^t e^{-k(t-u)}dW(u)\right)^2\right] &= \int_s^t e^{-2k(t-u)}du \\
&= \left[\frac{e^{2ku-2kt}}{2k}\right]_{u=s}^{u=t} \\
&= \frac{1-e^{-2k(t-s)}}{2k}
\end{aligned}
$$

Hence, conditional on$\mathcal{F}_s$, $r(t)$ is normally distributed with mean and variance given by:

$$
\begin{aligned}
\mathbb{E}[r(t)|\mathcal{F}_s] &= r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) \\
\text{Var}[r(t)|\mathcal{F}_s] &= \frac{\sigma^2}{2k}[1 - e^{-2k(t-s)}]
\end{aligned}
$$

This implies that, for each time t, the rate $r(t)$ can be negative with positive probability.

Note that, when $t \to \infty$, the expected rate tends to the value θ. The fact that θ can be regarded as a long-term average could also be inferred from the dynamics of the short-rate process itself. Notice, that the drift of the process $(r(t), t \in [0, \infty))$ is positive whenever the short rate is below θ and negative otherwise, so that r is pushed, at every time, to be closer on average to the level θ.

Term Structure and Bond Price Dynamics.

The short rate under the Vasicek model was shown to be:

$$r_t = r_s e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)} dW(u)$$

Now integrating the short rate from s to T, we get:

$$\begin{aligned}
\int_s^T r_t dt &= r_s \int_s^T e^{-k(t-s)} dt + \theta \int_s^T (1 - e^{-k(t-s)}) dt + \sigma \int_s^T \int_u^T e^{-k(t-u)} dW(u) dt \\
&= r_s \left[\frac{e^{-k(t-s)}}{-k} \right]_{t=s}^{t=T} + \theta(T-s) - \theta \left[\frac{e^{-k(t-s)}}{-k} \right]_{t=s}^{t=T} + \sigma \int_s^T \int_u^T e^{-k(t-u)} dt dW(u) \\
&= r_s \left(\frac{e^{-k(T-s)} - 1}{-k} \right) + \theta(T-s) - \theta \left(\frac{e^{-k(T-s)} - 1}{-k} \right) + \sigma \int_s^T \left[\frac{e^{-k(t-u)}}{-k} \right]_{t=u}^{t=T} dW(u) \\
&= r_s \left(\frac{1 - e^{-k(T-s)}}{k} \right) + \theta \left[(T-s) - \frac{1 - e^{-k(T-s)}}{k} \right] + \frac{\sigma}{k} \int_s^T \left(1 - e^{-k(T-u)} \right) dW(u)
\end{aligned}$$

Again, the Wiener integral term $\int_s^T \left(1 - e^{-k(T-u)}\right) dW(u)$ is Gaussian with mean 0 and variance:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_s^T \left(1 - e^{-k(T-u)} \right) dW(u) \right)^2 \right] &= \int_s^T \left(1 - e^{-k(T-u)} \right)^2 du \\
&= \int_s^T (1 - 2e^{-k(T-u)} + e^{-2k(T-u)}) du \\
&= \left[u - 2\frac{e^{-k(T-u)}}{k} + \frac{e^{-2k(T-u)}}{2k} \right]_{u=s}^{u=T} \\
&= (T - \frac{2}{k} + \frac{1}{2k} - (s - \frac{2}{k} e^{-k(T-s)} + \frac{e^{-2k(T-s)}}{2k})) \\
&= (T-s) - \frac{2}{k}(1 - e^{-k(T-s)}) + \frac{1}{2k}(1 - e^{-2k(T-s)}) \\
&= \frac{1}{2k}(2k(T-s) - 4(1 - e^{-k(T-s)}) + (1 - e^{-2k(T-s)})) \\
&= \frac{1}{2k}(2k(T-s) - 3 + 4e^{-k(T-s)} - e^{-2k(T-s)}))
\end{aligned}$$

Hence, conditional on $\mathcal{F}_s$, $\int_s^T r_t dt$ is normally distributed with mean and variance:

$$\begin{aligned}
\mathbb{E}[\int_s^T r_t dt | \mathcal{F}_s] &= r_s \left(\frac{1 - e^{-k(T-s)}}{k} \right) + \theta \left[(T-s) - \frac{1 - e^{-k(T-s)}}{k} \right] \\
\text{Var}[\int_s^T r_t dt | \mathcal{F}_s] &= \frac{\sigma^2}{2k^3}(2k(T-s) - 3 + 4e^{-k(T-s)} - e^{-2k(T-s)}))
\end{aligned}$$

Hence, the bond price at time s can be represented as:

$$P(s,T) = \mathbb{E}\left[e^{-\int_s^T r_t dt}|\mathcal{F}_s\right] = \exp\left[-\mathbb{E}[\int_s^T r_t dt|\mathcal{F}_s] + \frac{1}{2}\text{Var}[\int_s^T r_t dt|\mathcal{F}_s]\right]$$
$$= \exp\left[-r_s\left(\frac{1-e^{-k(T-s)}}{k}\right) - \theta\left((T-s) - \frac{1-e^{-k(T-s)}}{k}\right) + \frac{\sigma^2}{4k^3}(2k(T-s) - 3 + 4e^{-k(T-s)} - e^{-2k(T\right.$$